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Characterization of Local Strict Convexity Preserving Interpolation Methods by C^1 Functions

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This paper is concerned with the problem of strict convexity preserving interpolation in one variable. It is shown that a strictly convex Hermite interpolant to strictly convex data can always be chosen smooth and even to be a polynomial. Furthermore, two-point Hermite strict convexity preserving interpolation schemes using neither tension parameters nor additional knots are classified according to a number of certain desirable properties like symmetry, quadratic exactness, affine invariance, etc. One of the main results is the characterization of the set of such methods as a one-parameter family of solutions to certain boundary value problems.

1. INTRODUCTION

Shape preserving and especially convexity preserving approximation and interpolation has become an issue of increasing importance during the past few years. While most of the attention seems to be paid now to multivariate problems, we wish to address here some univariate aspects which we find of interest in their own right but which may also be of some help for systematic multivariate investigations.

The existing numerous univariate methods may be roughly divided into two groups. One of the first papers addressing shape preserving requirements is due to Schweikert in 1966 [15]. He introduced the idea of tension, an extra parameter whose adequate choice produces convexity

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preserving interpolants. After Schweikert's work, many type of tensioned interpolants were described (see the examples mentioned in [3]). Alternatively, in [6] and later in [14] piecewise quadratic polynomials were used achieving C^1 continuity to obtain convexity preserving interpolants by introducing additional knots. This knot insertion can be viewed as providing a new free parameter, i.e., as a way of applying tension (see [3]). For further information on piecewise polynomial convexity preserving interpolation see [4, 5, 9, 10].

In this paper we study and classify methods which neither introduce new knots nor tension parameters. One example of such methods was considered by Schaback [12] in 1973. He studied interpolation with C^2 rational splines of the form

$$f\,|_{[x_i, x_{i+1}]}(x) = \frac{a_i + b_i x + c_i x^2}{d_i + e_i x}.$$

This method being similar to interpolation by cubic splines preserves convexity, but requires nonlinear procedures to determine the interpolants.

This paper is organized as follows. In Section 2, we define the meaning of (strictly) convex data and prove that strictly convex data always admit strictly convex interpolants of arbitrary smoothness.

In Section 3, some desirable properties of local two-point strict-convexity preserving interpolation methods are compiled and discussed. These properties provide useful tools for the classification of all the known methods of (strict) convexity preserving interpolation. Tension methods can be included if the tension parameter is data dependent. It is also shown that those methods which depend continuously on the data are necessarily nonlinear.

Finaly, Section 4 and Section 5 are devoted to a complete characterization of all these methods satisfying the requirements mentioned in Section 3 by means of a family of boundary value problems for certain ordinary differential equations.

2. Strictly Convexity Preserving Interpolation by Smooth Functions

Aside from establishing existence of convexity preserving interpolation schemes an important issue is to select among all such schemes a particular one which has certain additional desirable properties. However, if one insists on smoothness as such an additional requirement one may encounter data which admit convex but not smooth convex interpolation.

For instance, the unique convex interpolant to the data $(j, |j|)$, $j = -2, -1, 0, 1, 2$ on the interval $[-2, 2]$ is given by $f(x) = |x|$ which is not everywhere differentiable.

In order to avoid such complications we will restrict our attention to strictly convex data.

DEFINITION 2.1. A set of data is said to be strictly convex, if there exists a strictly convex interpolant to these data.

Given $a = x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n = b \in \mathbb{R}$ with $x_i < x_{i+2}$, $i = 1, ..., n-2$, and strictly convex data $f_i \in \mathbb{R}$, we wish to find a strictly convex continuously differentiable function f satisfying $\lambda_i(f) = f_i$, where

$$\lambda_i(f) = \begin{cases} f'(x_i) & \text{if } \quad x_{i-1} = x_i, \\ f(x_i) & \text{otherwise.} \end{cases}$$

Defining

$$m_{i,i+1} := \begin{cases} f_{i+1} & \text{if } \quad x_{i+1} = x_i, \\ \dfrac{f_{i+1} - f_i}{x_{i+1} - x_i} & \text{otherwise,} \end{cases} \tag{2.1}$$

a necessary condition for the data to be strictly convex is that

$$m_{12} < m_{23} < \cdots < m_{n-1,n}. \tag{2.2}$$

In Theorem 2.2, we show that this condition is also sufficient to ensure the existence of a strictly convex smooth interpolant.

It is always possible to introduce additional data which are compatible with condition (2.2), and thus every solution of the extended interpolation problem is, in particular, a solution of the original problem.

To be more specific, let x_i be a single point in the sequence

$$a = x_1 \leqslant \cdots \leqslant x_{i-1} < x_i < x_{i+1} \leqslant \cdots \leqslant x_n = b.$$

Making x_i a double point

$$a = x_1 \leqslant \cdots \leqslant x_{i-1} < x_i = x_i < x_{i+1} \leqslant \cdots \leqslant x_n = b,$$

one may, for instance, choose $(m_{i-1,i} + m_{i,i+1})/2$ as the value for $f'(x_i)$. Likewise, choosing appropriate values for the slopes at the end points of the interval if necessary, we may in the following assume without loss of generality that the given data set is complete which means that function values and slopes are prescribed at each point.

THEOREM 2.2. *Given any point $x_1 < x_2 < \cdots < x_n$ in the closed interval $[a, b]$ and $y_i, m_i \in \mathbb{R}$, $i = 1, ..., n$ such that*

$$m_i < \frac{y_{i+1} - y_i}{x_{i+1} - x_i} < m_{i+1}, \qquad i = 1, ..., n-1, \tag{2.3}$$

there exists a strictly convex polynomial P such that

$$P(x_i) = y_i, \qquad i = 1, ..., n,$$
$$P'(x_i) = m_i, \qquad i = 1, ..., n.$$

More precisely, P can be chosen to satisfy

$$P''(x) \geqslant \tau, \qquad x \in [a, b], \tag{2.4}$$

for some $\tau > 0$.

Proof. The first step is to construct for some $k \geqslant 2$ a strictly convex C^k interpolant f satisfying $f''(x) > 0$, $x \in [a, b]$. To this end, let

$$\varepsilon = \min_{i=1,2,...,n-1} \left\{ \frac{y_{i+1} - y_i - m_i(x_{i+1} - x_i)}{(x_{i+1} - x_i)^2}, \frac{m_{i+1}(x_{i+1} - x_i) - y_{i+1} + y_i}{(x_{i+1} - x_i)^2} \right\}, \tag{2.5}$$

which is a positive constant by (2.3) and define

$$\hat{y}_i := y_i - \frac{\varepsilon}{2} x_i^2,$$
$$\hat{m}_i := m_i - \varepsilon x_i,$$

so that $\hat{y}_i, \hat{m}_i \in \mathbb{R}$, $i = 1, ..., n$ are perturbed values and slopes still satisfying (2.5) and the analogous quantity defined by (2.5) is $\varepsilon/2$. Let

$$L_i(x) := \hat{y}_i + \hat{m}_i(x - x_i),$$

and

$$s(x) := \max_{i=1,...,n} L_i(x),$$

then s is convex, piecewise linear and has no breakpoints at the x_i.

Now let k be an arbitrary but fixed positive integer. If N is sufficiently large, depending on k and the data, the Nth order Bernstein polynomials

$$B_{N,i}(s)(x) := \sum_{j=0}^{N} s\left(x_i + \frac{j(x_{i+1} - x_i)}{N}\right) \binom{N}{j} \left(\frac{x - x_i}{x_{i+1} - x_i}\right)^j \cdot \left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)^{N-j}$$

satisfy

$$B_{N,i}(s)(x_l) = \hat{y}_l, \qquad B'_{N,i}(s)(x_l) = \hat{m}_l, \qquad B^{(r)}_{N,i}(s)(x_l) = 0,$$
$$r = 2, ..., k, \qquad l = i, i+1.$$

Hence the function $\hat{f}$ defined by

$$\hat{f}\,|_{[x_i,\, x_{i+1}]} = B_{N,\,i}(s)$$

has by construction continuous derivatives of order k and is convex because s is convex. Thus the function

$$f(x) := \hat{f}(x) + \tfrac{\varepsilon}{2}\, x^2$$

is a strictly convex function satisfying the interpolation conditions

$$\begin{aligned} f(x_i) &= y_i \\ f'(x_i) &= m_i \end{aligned} \qquad i = 1, ..., n.$$

Moreover,

$$f''(x) \geqslant \frac{\varepsilon}{2}, \qquad x \in [a, b].$$

Let $H_{2n-1}(f)$ be the unique polynomial of degree $2n-1$ satisfying

$$H_{2n-1}(f)(x_i) = y_i, \qquad H'_{2n-1}(f)(x_i) = m_i, \qquad i = 1, ..., n, \tag{2.6}$$

and define

$$g(x) := \frac{f(x) - H_{2n-1}(f)(x)}{\prod_{j=1}^{n} (x - x_j)^2} = [x_1, x_1, x_2, x_2, ..., x_n, x_n, x]\, f,$$

where the expression on the right hand side denotes the divided difference of f with respect to the points listed between the square brackets. Thus g is $C^2[a, b]$ provided that $k \geqslant 4$. Hence for every $\delta > 0$, there exists some v an a polynomial P_v of degree v on $[a, b]$ such that

$$\max_{x \in [a,\, b]} |g^{(l)}(x) - P_v^{(l)}(x)| \leqslant \delta, \qquad l = 0, 1, 2. \tag{2.7}$$

Setting

$$A_\delta(x) := H_{2n-1}(f)(x) + \prod_{j=1}^{n} (x - x_j)^2\, P_v(x),$$

we see that A_δ still satisfies the interpolation conditions (2.6), while

$$f''(x) - A''_\delta(x) = \frac{d^2}{dx^2}\left[\left(\prod_{j=1}^{n} (x - x_j)^2\right)(g(x) - P_v(x))\right].$$

Thus by (2.7)

$$\max_{x \in [a, b]} |f''(x) - A''_\delta(x)| \leqslant C\delta,$$

for some constant C independent of δ. Hence for δ sufficiently small we obtain

$$A''_\delta(x) \geqslant \frac{\varepsilon}{4} > 0, \qquad x \in [a, b].$$

Taking $P = A_\delta$ and $\tau = \varepsilon/4$, the result follows. ∎

The previous theorem shows that condition (2.2) characterizes strictly convex data. Furthermore, it shows that strict convexity preserving interpolants can be made arbitrarily smooth, even analytic so that they satisfy (2.4) for some $\tau > \varepsilon/4$, $\varepsilon > 0$ given by (2.5), which is clearly stronger than strict convexity.

Our proof can also be regarded as a Weierstraß theorem for strictly convex interpolants. The set of strictly convex polynomial interpolants is dense in the set of strictly convex interpolants with respect to the topology of uniform convergence.

3. Some Properties of Local Methods

In this section we shall restrict our attention to the problem of finding strictly convex C^1 interpolants. Let us denote by $K^1[a, b]$ the set of all strictly convex functions in $C^1[a, b]$.

As mentioned in the previous section, we may assume that the given set of data is complete. If the interpolant needs only to be K^1, each complete problem can be split into several two-point problems of the type

$$\begin{aligned} f(x_i) &= y_i, \qquad & f(x_{i+1}) &= y_{i+1}, \\ f'(x_i) &= m_i, \qquad & f'(x_{i+1}) &= m_{i+1}, \end{aligned} \qquad i = 0, ..., n-1, \tag{3.1}$$

and a solution may be described as a piecewise function such that in each subinterval $[x_i, x_{i+1}]$, $i = 0, ..., n-1$, coincides with a solution of each of the problems (3.1). That means that for each set of data the computation of an interpolant can be done in such a way that changing the data at x_i, $i = 0, ..., n$ produces changes only on the subintervals next to x_i. Therefore the problem of Hermite interpolation with K^1 functons can be solved by means of local methods.

Let us define the mapping

$$L_{x_0, x_1}: K^1[x_0, x_1] \to \mathbb{R}^2 \times \mathbb{R}^2_+$$

$$f \mapsto \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_1) - f(x_0) - f'(x_0)(x_1 - x_0) \\ f(x_0) - f(x_1) - f'(x_1)(x_0 - x_1) \end{pmatrix}, \tag{3.2}$$

which is surjective by the characterization of the strictly convex data obtained in the last section.

DEFINITION 3.1. A local strict convexity preserving method by C^1 functions, or simply a K^1-method is a collection of mappings

$$P(\cdot \mid x_0, x_1): \mathbb{R}^2 \times \mathbb{R}^2_+ \to K^1[x_0, x_1]$$
$$u \mapsto P(u \mid x_0, x_1)$$

defined for all $x_0 \neq x_1$ such that $L_{x_0, x_1}(P(u \mid x_0, x_1)) = u$, for all $u \in \mathbb{R}^2 \times \mathbb{R}^2_+$, where $[x_0, x_1]$ also denotes the closed interval $[x_1, x_0]$ for $x_0 > x_1$.

We proceed now to collect and discuss some desirable properties of two-point (local) Hermite interpolation schemes preserving strict convexity.

Symmetry

Since the interpolation conditions are identical for $P(u_0, u_1, u_2, u_3 \mid x_0, x_1)$ and for $P(u_1, u_0, u_3, u_2 \mid x_1, x_0)$, the methods should be symmetric, that is,

$$P(u_0, u_1, u_2, u_3 \mid x_1, x_0)(x) = P(u_1, u_0, u_3, u_2 \mid x_0, x_1)(x_1 + x_0 - x). \tag{3.3}$$

Continuity

The set $K^1[x_0, x_1]$ inherits the topology of the space $C^1[x_0, x_1]$, i.e., given $f_n, f \in K^1[x_0, x_1]$, we say that $f_n \to f$ in $K^1[x_0, x_1]$ when

$$f_n \to f \qquad \text{uniformly in } [x_0, x_1]$$

and

$$f'_n \to f' \qquad \text{uniformly in } [x_0, x_1].$$

DEFINITION 3.2. A K^1-method is said to be continuous if

$$P(\cdot \mid x_0, x_1): \mathbb{R}^2 \times \mathbb{R}^2_+ \to K^1[x_0, x_1]$$

is a continuous mapping for all $x_0 \neq x_1$.

THEOREM 3.3. *There exists no extended continuous mapping* $\bar{P}: \mathbb{R}^4 \to C^1[a, b]$, with $L_{a,b}(\bar{P}(u)) = u$, *for all* $u \in \mathbb{R}^4$, *such that* $\bar{P}(\mathbb{R}^2 \times \mathbb{R}^2_+) \subseteq K^1[a, b]$.

Proof. Let $u^n = (0, 0, 1/n, 1) \in \mathbb{R}^2 \times \mathbb{R}^2_+$ and $f_n = \bar{P}(u^n) \in K^1[a, b]$. The sequence u^n converges to $u = (0, 0, 0, 1) \notin \mathbb{R}^2 \times \mathbb{R}^2_+$. Let f be $\bar{P}(u) \in C^1[a, b]$.

Suppose that $\bar{P}$ is continuous. Then $f_n \to f$ in $C^1[a, b]$. The functions f_n are convex and

$$f_n(a) = 0, \qquad f_n(b) = 0,$$

$$f'_n(a) = \frac{-1}{b-a}, \qquad f'_n(b) = \frac{1}{n(b-a)}.$$

Hence, we have

$$f_n(x) \leqslant 0, \qquad f_n(x) \geqslant \frac{-1}{n(b-a)}(x-a), \qquad f_n(x) \geqslant \frac{1}{b-a}(x-b).$$

However, the functions

$$\varphi_n(x) := \begin{cases} \dfrac{-1}{n} \cdot \dfrac{x-a}{b-a} & \text{if } x \in \left[a, \dfrac{a+nb}{n+1}\right] \\ -\dfrac{b-x}{b-a} & \text{if } x \in \left[\dfrac{a+nb}{n+1}, b\right] \end{cases}.$$

are negative, continuous, and converge uniformly to 0. Since $\varphi_n \leqslant f_n \leqslant 0$, f_n converges to zero uniformly as n tends to infinity. Furthermore, since $f_n \to f$ in $C^1[a, b]$ we conclude that $f = 0$ and thus

$$L_{a,b}(f) = (0, 0, 0, 0),$$

which contradicts that $f = \bar{P}(0, 0, 0, 1)$. This proves that such a mapping cannot exist (see Fig. 3.1). ∎

An interesting consequence of Theorem 3.3 is that K^1-methods cannot be linear.

COROLLARY 3.4. *There exists no linear* K^1*-method.*

Proof. If for some $x_0 \neq x_1$ the mapping $P(\cdot \mid x_0, x_1)$ were linear, we would be able to extend it by linearity to

$$\bar{P}: \mathbb{R}^4 \to C^1[x_0, x_1].$$

Since $\mathbb{R}^4$ is a finite dimensional vector space, $\bar{P}$ must be continuous, which contradicts Theorem 3.3. ∎

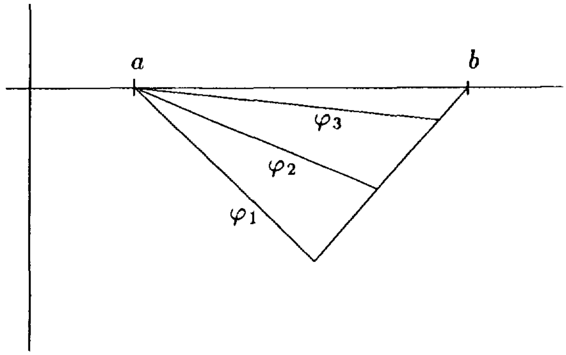

FIGURE 3.1

Change of Interval

Affine changes of variables take convex functions into convex functions on some other interval. Thus the question arises whether a K^1-method is invariant under affine transformations of the domain.

DEFINITION 3.5. A K^1-method $P(\cdot \mid x_0, x_1)$ is affinely invariant if

$$P(u \mid Ax_0, Ax_1)(Ax) = P(u \mid x_0, x_1)(x) \tag{3.4}$$

for all bijective affine maps A: $\mathbb{R} \to \mathbb{R}$.

From (3.4) we obtain the following formula of change of interval:

$$P(u \mid x_0, x_1)(x) = P(u \mid \xi_0, \xi_1)\left(\frac{x_1 - x}{x_1 - x_0}\,\xi_0 + \frac{x - x_0}{x_1 - x_0}\,\xi_1\right). \tag{3.5}$$

Using the symmetry property (3.3), we also derive the following formula of inversion of the interval:

$$P(u_1, u_0, u_3, u_2 \mid x_0, x_1)(x) = P(u_0, u_1, u_2, u_3 \mid x_0, x_1)(x_0 + x_1 - x). \tag{3.6}$$

Homogeneity of the Method

Clearly a function f is strictly convex if and only if tf is strictly convex for any $t \in \mathbb{R}^+$.

DEFINITION 3.6. A K^1-method $P(\cdot \mid x_0, x_1)$ is said to be homogeneous if

$$P(tu \mid x_0, x_1) = tP(u \mid x_0, x_1), \qquad \text{for all} \quad t \in \mathbb{R}^+. \tag{3.7}$$

Invariance under Addition of Affine Functions

Obviously $K^1[x_0, x_1]$ is invariant under addition of affine functions.

DEFINITION 3.7. A K^1-method $P(\cdot|x_0, x_1)$ is invariant under addition of affine functions, if

$$P(u + L_{x_0,x_1} g | x_0, x_1) = P(u|x_0, x_1) + g. \tag{3.8}$$

for all affine functions g.

From the previous definition follows that any method which is invariant under addition of affine functions satisfies

$$P(u_0 + v_0, u_1 + v_1, u_2, u_3 | x_0, x_1) = P(u_0, u_1, u_2, u_3 | x_0, x_1)(x) + \frac{x_1 - x}{x_1 - x_0} v_0 + \frac{x - x_0}{x_1 - x_0} v_1. \tag{3.9}$$

Quadratic Precision

Strictly convex quadratic functions are the simplest functions in $K^1[x_0, x_1]$. We use the notation

$$\Pi_2^+[x_0, x_1] = K^1[x_0, x_1] \cap \Pi_2[x_0, x_1], \tag{3.10}$$

where $\Pi_2[x_0, x_1]$ denotes the vector space of polynomials of degree less than or equal to 2 on $[x_0, x_1]$. From the fact that the second derivatives of quadratic functions are constant, one easily derives that

$$L_{x_0,x_1}(\Pi_2^+[x_0, x_1]) = \{(u_0, u_1, u_2, u_3) \in \mathbb{R}^2 \times \mathbb{R}_+^2 \mid u_2 = u_3\}.$$

DEFINITION 3.8. A K^1-method $P(\cdot|x_0, x_1)$ is said to have quadratic precision, if

$$P(L_{x_0,x_1}(q) | x_0, x_1) = q \qquad \text{for all} \quad q \in \Pi_2^+[x_0, x_1], \tag{3.11}$$

that is, $P(u_0, u_1, u_2, u_3 | x_0, x_1) \in \Pi_2^+[x_0, x_1]$ if and only if $u_2 = u_3$

If a K^1-method $P(\cdot|x_0, x_1)$ has quadratic precision then we have

$$P(u_0, u_1, u, u) = u_0 \cdot \lambda_0(x) + u_1 \cdot \lambda_1(x) - u \cdot \lambda_0(x)\, \lambda_1(x),$$

where λ_0, λ_1 are the barycentric coordinates of x with respect to the interval $[x_0, x_1]$

$$\lambda_0(x) = \frac{x_1 - x}{x_1 - x_0}, \qquad \lambda_1(x) = \frac{x - x_0}{x_1 - x_0}. \tag{3.12}$$

Restriction Property

It would be a desirable feature of a method, if the restriction of a solution to a smaller interval were a solution on this interval,

$$P(u|x_0, x_1)|_{[\xi_0,\xi_1]} = P(L_{\xi_0,\xi_1}[P(u|x_0, x_1)] \mid \xi_0, \xi_1), \tag{3.13}$$

for all $[\xi_0, \xi_1] \subseteq [x_0, x_1]$. This restriction property makes it possible to construct interpolants by subdivision. More precisely if we were able to evaluate $P(u \mid x_0, x_1)$ and its derivative at some point $x_{1/2} \in (x_0, x_1)$, then the interpolant on $[x_0, x_1]$ could be obtained by solving to subproblems on the intervals $[x_0, x_{1/2}]$ and $[x_{1/2}, x_1]$.

We can ask now the inverse question. Do there exist extended solutions?

DEFINITION 3.9. A solution of a K^1-method $P(\cdot \mid x_0, x_1)$ is a strictly convex function $f \in C^1(I)$ for some interval I, such that for each $x_0 \neq x_1 \in I$, we have

$$f\mid_{[x_0, x_1]} = P(L_{x_0, x_1}(f) \mid x_0, x_1).$$

If $P(\cdot \mid x_0, x_1)$ satisfies the restriction property, then every function of the form $P(u \mid x_0, x_1)$ is a solution and the restriction of any solution to a smaller interval is also a solution.

DEFINITION 3.10. Let $f \in C^1(I)$ be a solution of a K^1-method, a solution $g \in C^1(J)$ is said to be an extension of f if $I \subseteq J$ and $g\mid_I = f$. We write $f \preccurlyeq g$ to express that g is an extension of f.

DEFINITION 3.11. Let $f \in C^1(I)$ be a solution. The solution f is said to be maximal, if there exists no extension of f except f itself.

If the restriction property holds it can easily be proved using Zorn's lemma that for every solution $f: I \to \mathbb{R}$, there exists a maximal solution $g: J \to \mathbb{R}$ such that $f \preccurlyeq g$.

The restriction property plays a central role in the following section.

4. DIFFERENTIAL EQUATIONS AND K^1-METHODS

The following set

$$B := \{ax + b - \sqrt{cx + d} \mid a, b, c, d \in \mathbb{R}, c \neq 0\} \cup \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}, a > 0\}, \tag{4.1}$$

of strictly convex functions defined on intervals, whose graph is contained in a proper parabola, and the set

$$R := \left\{ax + b + \frac{1}{cx + d} \,\middle|\, a, b, c, d \in \mathbb{R}, c \neq 0\right\} \cup \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}, a > 0\} \tag{4.2}$$

of strictly convex functions defined on intervals, whose graph is contained in a hyperbola with a vertical asymptote or in a parabola with vertical axis will play an important role in the subsequent analysis. To this end, observe that

$$B(u \mid x_0, x_1) = u_0\lambda_0(x) + u_1\lambda_1(x) \\ -2\lambda_0(x)\,\lambda_1(x) \Big/ \left(\frac{\lambda_0(x)}{u_2} + \frac{\lambda_1(x)}{u_3} + \sqrt{\frac{\lambda_0(x)}{u_2^2} + \frac{\lambda_1(x)}{u_3^2}}\right) \quad (4.3)$$

and

$$R(u \mid x_0, x_1)(x) = u_0\lambda_0(x) + u_1\lambda_1(x) - \lambda_0(x)\lambda_1(x) \Big/ \left(\frac{\lambda_0(x)}{u_2} + \frac{\lambda_1(x)}{u_3}\right), \quad (4.4)$$

where $\lambda_0(x)$, $\lambda_1(x)$ denote the barycentric coordinates of x with respect to the points x_0, x_1 defined in (3.13), are the unique solutions of the strictly convex interpolation problem $L_{x_0, x_1}(f) = u$, $u \in \mathbb{R}^2 \times \mathbb{R}_+^2$ in the sets B and R, respectively.

We sketch here only the proof of the case (4.3). The proof of (4.4) is completely analogous and is left to the reader (see the paper by Schaback [12]).

Let $f(x)$ be an interpolant in B. Assuming first that $u_2 \neq u_3$, let

$$g(x) := f(x) - (u_0\lambda_0(x) + u_1\lambda_1(x)), \quad (4.5)$$

so that $L_{x_0, x_1}(g) = (0, 0, u_2, u_3)$. It is easy to see that g must be of the form

$$g(x) = a\lambda_0(x) + b\lambda_1(x) - \sqrt{a^2\lambda_0(x) + b^2\lambda_1(x)}. \quad (4.6)$$

Now upon differentiating, we obtain

$$g'(x) = \frac{b-a}{x_1 - x_0}\left[1 - \frac{b+a}{2\sqrt{a^2\lambda_0(x) + b^2\lambda_1(x)}}\right],$$

that is,

$$g'(x_0) = \frac{-(b-a)^2}{2a(x_1 - x_0)} = \frac{-u_2}{x_1 - x_0}$$

$$g'(x_1) = \frac{(b-a)^2}{2b(x_1 - x_0)} = \frac{u_1}{x_1 - x_0}.$$

This provides

$$a = \frac{2u_2u_3^2}{(u_3 - u_2)^2}, \qquad b = \frac{2u_2^2u_3}{(u_3 - u_2)^2}, \quad (4.7)$$

proving the uniqueness of the interpolant. Inserting these values into (4.7), provides

$$g(x)=\left[2\Big/\left(\frac{1}{u_2}-\frac{1}{u_3}\right)^2\right]$$
$$\times\left[\frac{\lambda_0(x)}{u_2}+\frac{\lambda_1(x)}{u_3}-\sqrt{\frac{\lambda_0(x)}{u_2^2}+\frac{\lambda_1(x)}{u_3^2}}\right]. \tag{4.8}$$

Multiplying and dividing by the conjugate

$$\frac{\lambda_0(x)}{u_2}+\frac{\lambda_1(x)}{u_3}+\sqrt{\frac{\lambda_0(x)}{u_2^2}+\frac{\lambda_1(x)}{u_3^2}},$$

we obtain an expression which is also valid for the case $u_2=u_3$

$$g(x)=-2\lambda_0(x)\,\lambda_1(x)\Big/\left(\frac{\lambda_0(x)}{u_2}+\frac{\lambda_1(x)}{u_3}+\sqrt{\frac{\lambda_0(x)}{u_2^2}+\frac{\lambda_1(x)}{u_3^2}}\right), \tag{4.9}$$

so that formula (4.3) is readily obtained.

Taking convex combinations of the formulas (4.3) and (4.4), we obtain the intermediate K^1-methods

$$P_w(\cdot\,|\,x_0,x_1):=(1-w)\,B(\cdot\,|\,x_0,x_1)+wR(\cdot\,|\,x_0,x_1), \qquad w\in[0,1], \tag{4.10}$$

covering (4.3) and (4.4) as extreme cases.

From formulae (4.3) and (4.4) we readily derive additional information on $B(\cdot\,|\,x_0,x_1)$ and $R(\cdot\,|\,x_0,x_1)$, namely, that both methods are continuous, affinely invariant, symmetric, invariant under addition of affine functions, homogeneous, and have quadratic precision.

These properties are also inherited by the intermediate methods $P_w(\cdot\,|\,x_0,x_1)$. However, the methods $B(\cdot\,|\,x_0,x_1)$ and $R(\cdot\,|\,x_0,x_1)$ have the additional feature that the restriction property holds. In both cases, the sets of functions B and R play the role of the set of maximal solutions whereas the mixed methods $P_w(\cdot\,|\,x_0,x_1)$, $w\in(0,1)$ do not share the restriction property. This indicates that it is the restriction property which is hard to realize for a K^1-method.

To shed some further light on this observation it is important to note that the sets B of (4.1) and R of (4.2) can be characterized by a property of the second derivative, namely,

$$B=\{f\,|\,f''(x)=(cx+d)^{-3/2},\,(c,d)\neq(0,0)\} \tag{4.11}$$

$$R=\{f\,|\,f''(x)=(cx+d)^{-3},\,(c,d)\neq(0,0)\}. \tag{4.12}$$

If $f \in B$ (resp. $f \in R$), then $(f'')^{1-\lambda}$ is linear for $\lambda = 5/3$ (resp. $\lambda = 4/3$), and f is a solution of

$$y^{IV} = \lambda \frac{(y''')^2}{y''}, \qquad y'' > 0. \tag{4.13}$$

This suggests the following considerations. Let

$$F: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$$
$$(x, y, y', y'', y''') \mapsto F(x, y, y', y'', y''')$$

be a continuous function such that the boundary value problem given by

$$\begin{aligned} y^{IV}(x) &= F(x, y(x), y'(x), y''(x), y'''(x)) \\ L_{x_0, x_1}(y) &= u, \qquad u \in \mathbb{R}^2 \times \mathbb{R}^2_+, \end{aligned} \tag{4.15}$$

has always a unique solution in $K^1[x_0, x_1]$. Then the solutions of these boundary value problems generate a K^1-method $P_F(u \mid x_0, x_1)$, which automaticaly satisfies the restriction property. Our next result shows that, under some additional conditions on the method, F has a special form.

THEOREM 4.1. *Suppose the fourth order differential equation*

$$y^{IV} = F(x, y, y', y'', y'''), \tag{4.16}$$

where $F \in C(\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R})$ defines a K^1-method with the properties of affine invariance, invariance under addition of affine functions and homogeneity. Then there exists a $\lambda \in \mathbb{R}$ such that F takes the form

$$F(x, y, y', y'', y''') = \lambda \frac{(y''')^2}{y''}. \tag{4.17}$$

Proof. By the existence theorem for ordinary differential equations we know that under the above assumptions on F, every initial value problem of the form

$$\begin{aligned} y^{IV} &= F(x, y, y', y'', y'''), \\ y(x_0) &= y_0, \\ y'(x_0) &= y'_0, \\ y''(x_0) &= y''_0 > 0, \\ y'''(x_0) &= y'''_0, \end{aligned} \tag{4.18}$$

always has a solution in a neighbourhood of x_0. Hence, for some closed interval $[a, b]$ containing x_0, we obtain a solution f: $[a, b] \to \mathbb{R}$, which is given by $P_F(L_{a,b}(f)|a, b)$. We will point out next how the required properties affect the structure of F.

We know that if $f \in C^4[a, b]$ is a solution of the differential equation and the method is invariant under affine transformations of the domain, then

$$f(\cdot - c): [a+c, b+c] \to \mathbb{R}$$
$$x \mapsto f(x-c)$$

is also a solution. This implies that

$$F(x_0, y_0, y_0', y_0'', y_0''') = f^{IV}(x_0) = F(x_0 + c, y_0, y_0', y_0'', y_0''').$$

Since $x_0, y_0, y_0', y_0'', y_0''', c$ are arbitrary we conclude that F is independent of its first argument. Hence, we can drop the argument x_0 from F.

If the method is invariant under addition of affine functions and if $f(x)$ is a solution, so is $f(x) + a + bx$. In particular, adding constant functions a, we derive

$$F(x_0, y_0, y_0', y_0'', y_0''') = f^{IV}(x_0) = F(x_0, y_0 + a, y_0', y_0'', y_0''')$$

and so F does not depend on y_0. Furthermore, adding $b(x - x_0)$, we conclude

$$F(x_0, y_0, y_0', y_0'', y_0''') = f^{IV}(x_0) = F(x_0, y_0, y_0' + b, y_0'', y_0''')$$

and F is seen to be independent of the variable y_0'. Hence the number of arguments of F may be further reduced to y_0'' and y_0'''.

By homogeneity $\lambda f(t)$ solves the problem for $\lambda y_0''$ and $\lambda y_0'''$, if f solves it for y_0'' and y_0''', which means that

$$F(\lambda y_0'', \lambda y_0''') = \lambda F(y_0'', y_0''') \qquad \text{for all} \quad \lambda \in \mathbb{R}. \tag{4.19}$$

The affine invariance with respect to the domain implies that if $f: I \to \mathbb{R}$ is a solution of the differential equation, then

$$f(r\cdot): \frac{1}{r} I \to \mathbb{R}$$
$$x \mapsto f(rx)$$

is also a solution which implies that

$$F(y_0'', y_0''') = \frac{1}{r^4} F(r^2 y_0'', r^3 y_0'''), \quad \text{for all} \quad r \neq 0. \tag{4.20}$$

If $y_0''' \neq 0$, taking $r = y_0''/y_0'''$ in (4.20) and applying (4.19), we deduce that

$$F(y_0'', y_0''') = \left(\frac{y_0'''}{y_0''}\right)^4 F\left(\frac{(y_0'')^3}{(y_0''')^2}, \frac{(y_0'')^3}{(y_0''')^2}\right) = \frac{(y_0''')^2}{y_0''} F(1, 1).$$

On the other hand, if $y_0''' = 0$, we can derive from the continuity of F that $F(y_0'', 0) = 0$. Then choosing $\lambda = F(1, 1)$ formula (4.17) is confirmed. ▮

So far we have imbedded the class of interesting K^1-methods into a one parameter family. In remains to identify the feasible parameters λ.

5. Boundary Value Problems with Feasible Parameters

In this section we shall characterize the values of $\lambda \in \mathbb{R}$ such that the boundary value problem

$$\begin{aligned} y^{IV} &= \lambda \cdot \frac{(y''')^2}{y''}, \qquad && y'' > 0, \\ L_{x_0, x_1}(y) &= u, && u \in \mathbb{R}^2 \times \mathbb{R}_+^2 \end{aligned} \tag{5.1}$$

defines a K^1-method.

In the sequel we make repeated use of the following simple observation.

Lemma 5.1. *Let $l_1(x)$, $l_2(x)$ be two increasing affine functions. Then*

(a) *there exist two constants $A, B \in \mathbb{R}$, $A > 0$ such that*

$$l_2(x) = A l_1(x + B).$$

(b) *There exist two constants $\bar{A}, \bar{B} \in \mathbb{R}$, $\bar{A} > 0$ such that*

$$l_2(x) = l_1(\bar{A}x) + \bar{B}.$$

Proof. The result follows immediately from straightforward calculations. ▮

Let us remark that a solution of (5.1) has a third derivative which is always positive, always zero or always negative. Now using this tricotomy and the Lemma 5.1, we can easily describe the set of maximal solutions of the differential equation in (5.1).

Theorem 5.2. *Let f_λ be a maximal solution of the differential equation*

$$y^{IV} = \lambda \cdot \frac{(y''')^2}{y''}, \qquad y'' > 0, \tag{5.2}$$

such that $f_\lambda''' \neq 0$, *then every other maximal solution* $g_\lambda(x)$ *is given by*

$$\text{(a)}\quad g_\lambda(x)=\begin{cases} af_\lambda(bx)+cx+d, & a, b>0, \quad \text{if } \lambda=1,\\ af_\lambda(x+b)+cx+d, & a>0, \quad \text{if } \lambda\neq 1,\end{cases} \tag{5.3}$$

if $g_\lambda'''(x)$ *has the same sign as* $f_\lambda'''(x)$,

$$\text{(b)}\quad g_\lambda(x)=ax^2+cx+d, \qquad a>0, \tag{5.4}$$

if $g_\lambda'''(x)=0$ *for all* x *and*

$$\text{(c)}\quad g_\lambda(x)=\begin{cases} af_\lambda(-bx)+cx+d, & a, b>0, \quad \text{if } \lambda=1,\\ af_\lambda(-x+b)+cx+d, & a>0, \quad \text{if } \lambda\neq 1,\end{cases} \tag{5.5}$$

if $g_\lambda'''(x)$ *has opposite sign to* $f_\lambda''(x)$.

Proof.

Case (a) Without loss of generality we can assume that $f_\lambda'''>0$ and therefore $g_\lambda'''>0$.

Suppose first that $\lambda=1$. Then the maximal solutions g_λ with $g_\lambda'''(x)>0$ are functions defined on the whole real line whose second derivatives are exponentials of increasing linear functions. Thus the functions $\log(g_\lambda'')$ and $\log(f_\lambda'')$ are increasing and linear. By Lemma 5.1, there are constant $\bar{A}>0$, $\bar{B}\in\mathbb{R}$ such that

$$\log g_\lambda''(x)=\log f_\lambda''(\bar{A}x)+\bar{B}.$$

Therefore, we have

$$g_\lambda''(x)=e^{\bar{B}}f_\lambda''(\bar{A}x)$$

i.e.,

$$\left[g_\lambda(x)-\frac{e^{\bar{B}}}{\bar{A}^2}f_\lambda(\bar{A}x)\right]''=0,$$

whence we conclude

$$g_\lambda(x)-\frac{e^{\bar{B}}}{\bar{A}^2}f_\lambda(\bar{A}x)=cx+d.$$

The assertion follows now with $a=e^{\bar{B}}/\bar{A}^2>0$ and $b=\bar{A}>0$.

Suppose now $\lambda\neq 1$, ten the maximal solutions g_λ with $g_\lambda'''>0$ are only defined at those points in which the linear function $(g_\lambda'')^{1/(1-\lambda)}$ is positive, i.e., f_λ and g_λ are defined on a half line. On the other hand,

$(1-\lambda)(g_\lambda'')^{1/(1-\lambda)}$ is an increasing linear function. Thus there exists $A>0$, $B\in\mathbb{R}$ such that

$$(1-\lambda)(g_\lambda'')^{1/(1-\lambda)}(x)=(1-\lambda)A\cdot(f_\lambda'')^{1/(1-\lambda)}(x+B).$$

Furthermore, $x\in\operatorname{Dom} g_\lambda$ if and only if $(g_\lambda'')^{1/(1-\lambda)}(x)>0$ which can be expressed in terms of f by $A(f_\lambda'')^{1/(1-\lambda)}(x+B)>0$, that is $x+B\in\operatorname{Dom} f_\lambda$.

Raising both sides of

$$(g_\lambda'')^{1/(1-\lambda)}(x)=A\cdot(f_\lambda'')^{1/(1-\lambda)}(x+B)$$

to the power $1-\lambda$ provides

$$g_\lambda-A^{1-\lambda}f_\lambda(x+B)=cx+d.$$

Taking $a=A^{1-\lambda}$, $b=B$ confirms formula (4.23).

Case (b) $g_\lambda'''=0$. In this case, the maximal solutions are the quadratic functions $q_\lambda(x)=ax^2+cx+d$, $a>0$.

Case (c) $g_\lambda'''(x)<0$. The function $g_\lambda(-\cdot)$ is a maximal solution belonging to the Case (a). Thus the problem can be reduced to Case (a). ∎

In view of Theorem 5.2, the problem of describing every maximal solution is reduced to finding only one maximal solution for every λ. In the following table we give a choice of a simple function f_λ with the property $f_\lambda'''>0$:

Values of λ	Solutions f_λ
$\lambda>2$	$f_\lambda:\mathbb{R}^-\to\mathbb{R}$, $x\mapsto(-x)^{2-(1/(\lambda-1))}$
$\lambda=2$	$f_\lambda:\mathbb{R}^-\to\mathbb{R}$, $x\mapsto(-x)\cdot\log(-x)$
$\frac{3}{2}<\lambda<2$	$f_\lambda:\mathbb{R}^-\to\mathbb{R}$, $x\mapsto-(-x)^{2-(1/(\lambda-1))}$
$\lambda=\frac{3}{2}$	$f_\lambda:\mathbb{R}\to\mathbb{R}$, $x\mapsto-\log(-x)$
$1<\lambda<\frac{3}{2}$	$f_\lambda:\mathbb{R}^-\to\mathbb{R}$, $x\mapsto\dfrac{1}{(-x)^{(1/(\lambda-1))-2}}$
$\lambda=1$	$f_\lambda:\mathbb{R}\to\mathbb{R}$, $x\mapsto e^x$
$\lambda<1$	$f_\lambda:\mathbb{R}^+\to\mathbb{R}$, $x\mapsto x^{2+(1/(1-\lambda))}$

Note that $\lambda = 5/3$ and $\lambda = 4/3$ corresponding to methods (4.3) and (4.4) are the only values such that the graphs of the solutions of the differential equations are conic sections.

The above table has a close relationship to the literature on nonlinear splines. The exponentials $\lambda = 1$ have a long history (see [15, 1, 8, 11] for more references). The functions with $\lambda = 4/3$ were studied in [12] for Lagrange interpolation. The other cases of the table occur in [13]. Minimization properties and computation of the nonlinear spline functions based upon piecewise solutions of (5.2) can be found in [2] (see also [7]).

Now let us return to the main objective of this section, namely, to determine whether for a given value of λ, the boundary value problem (5.1) defines a K^1-method satisfying the properties mentioned in Section 3. In view of the properties of the solutions of the differential equation (5.2), this question is equivalent to showing that the boundary value problem (5.1) has always a unique solution which depends continuously on u.

THEOREM 5.3. *Let $\psi_\lambda : D_\lambda \to (0, 1)$ be given by*

$$\psi_\lambda(x) := \begin{cases} \dfrac{[0, 0, x] f_\lambda}{[0, x, x] f_\lambda}, & \text{if } \lambda = 1 \\[2ex] \dfrac{[x, x, x+1] f_\lambda}{[x, x+1, x+1] f_\lambda}, & \text{if } \lambda \neq 1, \end{cases} \tag{5.6}$$

where

$$D_\lambda := \begin{cases} (-\infty, -1) & \text{if } \lambda > 1 \\ \mathbb{R}^+ & \text{if } \lambda \leqslant 1, \end{cases} \tag{5.7}$$

and f_λ is the function defined in the above table. Then

(a) *The boundary value problem* (5.1) *has always a solution if and only if ψ_λ is surjective.*

(b) *The boundary value problem* (5.1) *has no more than one solution* (*uniqueness*) *if and only if ψ_λ is injective.*

(c) *The boundary value problem* (5.1) *has always a unique solution which depends continuously on u in $K^1[x_0, x_1]$ if and only if ψ_λ is bijective and ψ_λ^{-1} is a continuous function.*

Proof. Note first that from the definition of ψ_λ and $f_\lambda''' > 0$ it follows that $\psi_\lambda(D_\lambda) \subseteq (0, 1)$.

Due to the invariance under changing the interval, we can use formula (3.5) to reduce our problem to the standard interval [0, 1]. Inverting the interval if necessary using formula (3.6) we may assume that $u_2 \leqslant u_3$.

Moreover, since we can add an adequate affine function, we can assume further that $u_0 = u_1 = 0$. By homogeneity, we can also rescale $u_2 \leqslant u_3$ so that $u_3 = 1$. Thus the problem is reduced to finding a function $g \in C^4[0, 1]$ such that

$$\begin{aligned} g^{IV} &= \lambda \cdot \frac{(g''')^2}{g''}, \qquad & g'' > 0, \\ L_{0,1}(g) &= (0, 0, u_2, 1), \qquad & u_2 \in (0, 1]. \end{aligned} \tag{5.8}$$

Let us analyze first the case $u_2 \neq 1$. Since g''' is always positive, always zero or always negative and $[0, 0, 1, 1]\, g = 1 - u_2 > 0$, we deduce that $g'''(x) > 0$, for all $x \in [0, 1]$. On the other hand the functions f_λ of the above table satisfy $f_\lambda''' > 0$. From Theorem 5.2, we obtain that

$$g(x) = \begin{cases} af_\lambda(bx) + cx + d, \quad a, b > 0, & \text{if } \lambda = 1, \\ af_\lambda(x + b) + cx + d, \quad a > 0, & \text{if } \lambda \neq 1. \end{cases} \tag{5.9}$$

Let us see first that b must be chosen in D_λ. If $\lambda = 1$, the condition $b > 0$ to ensure positivity of the third derivative is equivalent to $b \in D_\lambda$. If $\lambda \neq 1$, we must have that

$$\mathrm{Dom}(af_\lambda(x + b) + cx + d) \supseteq [0, 1],$$

which is equivalent to saying that $\mathrm{Dom}(f_\lambda) \supseteq [b, b + 1]$. If $\mathrm{Dom}(f_\lambda) = \mathbb{R}_+$, then b can be any value of $\mathbb{R}_+$. If $\mathrm{Dom}(f_\lambda) = \mathbb{R}_-$, then b must be chosen in the interval $(-\infty, -1)$. Imposing the interpolation conditions $g(0) = 0$, $g(1) = 0$, $[0, 1, 1]\, g = 1$, we deduce that g is of the form

$$g(x) = \begin{cases} \dfrac{1}{b^2 [0, b, b] f_\lambda} [f_\lambda(bx) - f_\lambda(0) + (f_\lambda(0) - f_\lambda(b))x], \\ \quad \text{if } \lambda = 1, \\ \dfrac{1}{[b, b+1, b+1] f_\lambda} [f_\lambda(x + b) - f_\lambda(b) + (f_\lambda(b) - f_\lambda(b + 1))x], \\ \quad \text{if } \lambda \neq 1, \end{cases} \tag{5.10}$$

where $b \in D_\lambda$. Now taking into account that $[0, 0, 1]\, g = \psi_\lambda(b)$, we deduce that $g(x)$ is a solution of the boundary value problem (5.8) if and only if $b \in D_\lambda$ is a solution of the nonlinear equation $\psi_\lambda(b) = u_2$, $u_2 \in (0, 1)$.

Let us analyze now the remaining case $u_2 = 1$. In this case the solution g satisfies $[0, 0, 1, 1]\, g = 0$. This implies that $g'''(x) = 0$ for all $x \in [0, 1]$ and then the unique solution of the boundary value problem (5.8) is the quadratic function $g(x) = x(x - 1)$.

From the above considerations (a) and (b) follow immediately.

For the proof of (c) recall that, when $u_2 \neq 1$, the unique solution of the boundary value problem (5.8) can be written in the form (5.10) with

$b=\psi_\lambda^{-1}(u_2)$. Thus the coefficient b depends continuously on u_2 if and only if the function g depends continuously on u_2 in $K^1[0,1]$. On the other hand, any solution g of the boundary value problem (5.8) with $u_2 \in (0,1)$ satisfies

$$1=[0,0,1]\,g \leqslant [0,1,x]\,g \leqslant [0,1,1]\,g = u_2$$

because $g''' > 0$. When $u_2 \to 1$, we have that $[0,1,x]\,g \to 1$ uniformly and thus $g(x) \to x(x-1)$ uniformly. From

$$1=[0,0,1]\,g \leqslant [x,x,1]\,g \leqslant [x,1,1]\,g \to 1 \text{ uniformly,}$$

we obtain that $g'(x) \to 1-2x$ uniformly. Thus continuity at $u_2=1$ follows. ▮

We will investigate next the properties of ψ_λ for the various ranges of λ. To settle the case $\lambda=1$ we prove

PROPOSITION 5.4. *The function ψ_1 is bijective and ψ_1^{-1} is a differentiable function.*

Proof. The function ψ_1 given by

$$\psi_1(x)=\frac{e^x-1-x}{1-(e^x-xe^x)}, \qquad x>0,$$

is differentiable and

$$\psi_1'(x)=\frac{x^2e^x-(e^x-1)^2}{(xe^x-e^x+1)^2}<0.$$

Therefore it is injective and ψ_1^{-1} is differentiable. Furthermore

$$\lim_{x\to 0}\psi_1(x)=1, \qquad \lim_{x\to +\infty}\psi_1(x)=0$$

and thus ψ_1 is onto $(0,1)$. ▮

LEMMA 5.5. *If $\lambda \neq 1$, then $\lim_{|x|\to\infty}\psi_\lambda(x)=1$.*

Proof. When $\lambda \neq 1$,

$$\psi_\lambda(x)=\frac{[x,x,x+1]\,f_\lambda}{[x,x+1,x+1]\,f_\lambda}.$$

By hypothesis, we have $f_\lambda''' > 0$ and we conclude

$$[x,x,x]\,f_\lambda < [x,x,x+1]\,f_\lambda < [x,x+1,x+1]\,f_\lambda < [x+1,x+1,x+1]\,f_\lambda,$$

which implies

$$\frac{f''_\lambda(x)}{f''_\lambda(x+1)} < \psi_\lambda(x) < 1.$$

But $f''_\lambda(x) = c_\lambda \, |x|^{1/(1-\lambda)}$ for all $x \in \operatorname{Dom} f_\lambda$ and thus

$$\lim_{|x| \to \infty} \frac{f''_\lambda(x)}{f''_\lambda(x+1)} = \lim_{|x| \to \infty} \left| \frac{x}{x+1} \right|^{1/(1-\lambda)} = 1. \quad \blacksquare$$

To deal with the remaining cases we employ the following technical lemma.

LEMMA 5.6. *Let*

$$h_\alpha(x) := (\alpha+2) \frac{(x+1)^{\alpha+1} - x^{\alpha+1}}{x^{\alpha+2} + (\alpha+2)(x+1)^{\alpha+1} - (x+1)^{\alpha+2}}, \qquad x > 0. \tag{5.11}$$

Then one has

$$\begin{cases} h'_\alpha(x) < 0 & \text{if } \alpha < -2, \\ h'_\alpha(x) < 0 & \text{if } -2 < \alpha < -1, \\ h'_\alpha(x) < 0 & \text{if } -1 < \alpha < 0, \\ h'_\alpha(x) > 0 & \text{if } \alpha > 0. \end{cases} \tag{5.12}$$

Proof. If $\alpha = -2, -1, 0$, $h'_\alpha = 0$. Let us assume for the rest of the proof that $\alpha \neq -2, -1, 0$. After some straightforward calculations, we obtain

$$h'_\alpha(x) = (\alpha+2) \cdot \frac{[(x+1)^{\alpha+1} - x^{\alpha+1}]^2 - (\alpha+1)^2 x^\alpha (x+1)^\alpha}{[x^{\alpha+2} + (\alpha+2)(x+1)^{\alpha+1} - (x+1)^{\alpha+2}]^2}.$$

The sign of $h'_\alpha(x)$ is the same as the sign of

$$(\alpha+2) \left[\frac{(x+1)^{\alpha+1} - x^{\alpha+1}}{\alpha+1} - x^{\alpha/2}(x+1)^{\alpha/2} \right], \qquad x \in \mathbb{R}^+,$$

which has the same sign as

$$(\alpha+2) \left[\frac{(1+1/x)^{\alpha+1} - 1}{\alpha+1} - \frac{1}{x} \left(1 + \frac{1}{x} \right)^{\alpha/2} \right], \qquad x \in \mathbb{R}^+,$$

Let us define

$$g(t) := (\alpha+2) \left[\frac{(1+t)^{\alpha+1} - 1}{\alpha+1} - t(1+t)^{\alpha/2} \right], \qquad t \geqslant 0,$$

then $\operatorname{sign}(h'_\alpha(x)) = \operatorname{sign}(g(1/x))$. In order to analyze the sign of the function g we compute the derivative

$$g'(t) = (\alpha+2)(1+t)^{\alpha/2-1}\left[(1+t)^{\alpha/2+1} - 1 - \left(\frac{\alpha}{2}+1\right)t\right].$$

Finally defining

$$k(t) := \frac{(1+t)^{\alpha/2+1} - 1 - (\alpha/2+1)t}{\alpha/2+1},$$

we deduce that

$$\operatorname{sign}(g'(t)) = \operatorname{sign}(k(t)) = \operatorname{sign}(\alpha).$$

and since $g(0)=0$, we obtain $\operatorname{sign}(g(t)) = \operatorname{sign}(\alpha)$ for $t>0$. ∎

Now we are ready to prove

PROPOSITION 5.7. (a) *If* $\lambda > 2$, *then* ψ_λ *is decreasing and* $\psi_\lambda(-\infty, -1) = ((\lambda-2)/(\lambda-1), 1)$.

(b) *If* $\lambda < 1$, *then* ψ_λ *is increasing and* $\psi_\lambda(0, +\infty) = ((1-\lambda)/(2-\lambda), 1)$.

Proof. Let us consider first the case $\lambda < 1$. Then setting $\alpha := 1/(1-\lambda)$ (and therefore $\alpha > 0$) we obtain

$$\begin{aligned}\psi_\lambda(x) &= \frac{(x+1)^{\alpha+2} - x^{\alpha+2} - (\alpha+2)x^{\alpha+1}}{-(x+1)^{\alpha+2} + x^{\alpha+2} + (\alpha+2)(x+1)^{\alpha+1}} \\ &= (\alpha+2)\frac{(x+1)^{\alpha+1} - x^{\alpha+1}}{x^{\alpha+2} + (\alpha+2)(x+1)^{\alpha+1} - (x+1)^{\alpha+2}},\end{aligned}$$

i.e., $\psi_\lambda(x) = h_\alpha(x)$. By Lemma 5.6, this function is increasing.

Analogously, we obtain for $\lambda > 2$

$$\psi_\lambda(x) = \frac{1}{h_\alpha(-x-1)}, \qquad x \in (-\infty, -1),$$

where $\alpha = -1/(\lambda-1)$ varies in the range $(-1, 0)$. Again we infer from Lemma 5.6 that h_α is decreasing and so is ψ_λ.

Now, recall that

$$\psi_\lambda(x) = \frac{[x, x, x+1] f_\lambda}{[x, x+1, x+1] f_\lambda},$$

where we see from the above table that f_λ is a function which can be extended to a strictly convex $C^1(\mathbb{R})$ function

$$\bar{f}_\lambda(x) = |x|^{2+1/(1-\lambda)}.$$

Thus, we have for $\lambda < 1$

$$\lim_{x \to 0^+} \psi_\lambda(x) = \lim_{x \to 0^+} \frac{[x, x, x+1] f_\lambda}{[x, x+1, x+1] f_\lambda} = \frac{[0, 0, 1] \bar{f}_\lambda}{[0, 1, 1] \bar{f}_\lambda} = \frac{1-\lambda}{2-\lambda}.$$

Analogously, for $\lambda > 2$ we obtain

$$\lim_{x \to -1^-} \psi_\lambda(x) = \lim_{x \to -1^-} \frac{[x, x, x+1] f_\lambda}{[x, x+1, x+1] f_\lambda} = \frac{[-1, -1, 0] \bar{f}_\lambda}{[-1, 0, 0] \bar{f}_\lambda}$$
$$= \frac{[1, 1, 0] \bar{f}_\lambda}{[1, 0, 0] \bar{f}_\lambda} = \frac{\lambda - 2}{\lambda - 1}. \quad \blacksquare$$

In neither case, there can be general solution for the boundary value problem (5.1). For example, when $\lambda = 0$ we obtain the cubic Hermite interpolant and get

$$\psi_0(\mathbb{R}_+) = (\tfrac{1}{2}, 1), \tag{5.13}$$

which means that in order to obtain a strictly convex interpolant the quotient u_2/u_3 is only allowed to vary between 1/2 and 1, when $u_2 < u_3$.

LEMMA 5.8. *Let* $f\colon \mathbb{R}^- \to \mathbb{R} \in K^1(\mathbb{R}^-)$ *be a function such that*

$$\lim_{x \to 0^-} f'(x) = +\infty, \tag{5.14}$$

then

$$\lim_{x \to 0^-} \frac{f(x)}{f'(x)} = 0. \tag{5.15}$$

Proof. By the convexity of f, its derivative f' is an increasing function so that

$$f'(x_1) \cdot (x_2 - x_1) \leqslant f(x_2) - f(x_1) \leqslant f'(x_2) \cdot (x_2 - x_1),$$

for $x_2 > x_1$. Since $\lim_{x \to 0^-} f'(x) = +\infty$, $f'(x)$ is positive on some interval $(-\delta, 0)$, thus

$$\frac{f(x_1) + f'(x_1) \cdot (x_2 - x_1)}{f'(x_2)} \leqslant \frac{f(x_2)}{f'(x_2)} \leqslant \frac{f(x_1)}{f'(x_2)} + x_2 - x_1.$$

Passing to the limit in the previous inequalities, we obtain

$$0 \leqslant \liminf_{x_2 \to 0^-} \frac{f(x_2)}{f'(x_2)} \leqslant \limsup_{x_2 \to 0^-} \frac{f(x_2)}{f'(x_2)} \leqslant -x_1.$$

This inequality is valid for each $x_1 \in \mathbb{R}_-$ and so (5.15) holds. ∎

The following theorem is the main result of this section and describes for which values of λ a solution of the boundary value problem (5.1) always exists and depends continuously on u.

THEOREM 5.9. *ψ_λ is bijective if and only if $\lambda \in [1, 2]$. Furthermore, ψ_λ^{-1} is a differentiable function.*

Proof. The case $\lambda = 1$ was just proved in Proposition 5.4. If $\lambda \in (1, 2]$, $\lim_{x \to 0^-} f'_\lambda(x) = +\infty$, which by Lemma 5.8 implies that

$$\lim_{x \to 0^-} \frac{f_\lambda(x)}{f'_\lambda(x)} = 0.$$

Therefore

$$\begin{aligned}
\lim_{x \to -1^-} \psi_\lambda(x) &= \lim_{x \to -1^-} \frac{[x, x, x+1] f_\lambda}{[x, x+1, x+1] f_\lambda} \\
&= \lim_{x \to -1^-} \frac{f_\lambda(x+1) - f_\lambda(x) - f'_\lambda(x)}{f'_\lambda(x+1) - f_\lambda(x+1) - f_\lambda(x)} \\
&= \lim_{x \to 0^-} \left[\frac{f_\lambda(x)}{f'_\lambda(x)} - \frac{f_\lambda(x-1) + f'_\lambda(x+1)}{f'_\lambda(x)} \right] \Big/ \\
&\qquad \left[1 - \frac{f_\lambda(x)}{f'_\lambda(x)} - \frac{f_\lambda(x+1)}{f'_\lambda(x)} \right] = 0,
\end{aligned}$$

and, by Lemma 5.5, ψ_λ is surjective. It remains to confirm that $\psi'_\lambda < 0$, which implies that ψ_λ is a decreasing function and that ψ_λ^{-1} is differentiable.

For $\lambda \in (1, \frac{3}{2})$ and $\lambda \in (\frac{3}{2}, 2)$ we have

$$\psi_\lambda(x) = \frac{[x, x, x+1] f_\lambda}{[x, x+1, x+1] f_\lambda} = \frac{1}{h_\alpha(-1-x)}, \qquad \text{with} \quad \alpha = \frac{-1}{\lambda - 1},$$

where h_α is given by (5.11). Since $\alpha \in (-\infty, -2) \cup (-2, -1)$, Lemma 5.6 implies that $\psi'_\lambda < 0$.

The only remaining cases are $\lambda = \frac{3}{2}$ and $\lambda = 2$. To prove that $\psi'_{3/2} < 0$ we define a new auxiliary function $h\colon \mathbb{R}^+ \to \mathbb{R}$ given by

$$h(x) = 1 + \frac{1}{\psi_{3/2}(-1-x)} = 1 + \left[\log\left(\frac{x+1}{x}\right) - \frac{1}{x}\right] \Big/ \left[\frac{1}{x+1} - \log\left(\frac{x+1}{x}\right)\right]$$

$$= \frac{1}{x(x+1)\log((x+1)/x) - x},$$

$$h'(x) = \frac{-(2x+1)\log((x+1)/x + 2}{(x(x+1)\log((x+1)/x) - x)^2}$$

$$= \frac{-(2x+1)[\log(1+x^{-1}) - (2x^{-1}/(2+x^{-1}))]}{(x(x+1)\log(1+x^{-1}) - x)^2}.$$

Moreover, defining the auxiliary function

$$g(s) := \log(1+s) - \frac{2s}{s+2}, \qquad s \geqslant 0,$$

the sign of $h'(x)$ coincides with the sign of $-g(1/x)$. Since

$$g'(s) = \frac{1}{s+1} - \frac{4}{(s+2)^2} = \frac{s^2}{(s+1)(s+2)^2} > 0$$

and $g(0) = 0$, we conclude $g(s) > 0$ for all $s \in \mathbb{R}^+$, which implies that $h'(x) < 0$ for all $x \in \mathbb{R}^+$ and therefore $\psi'_\lambda(x) < 0$ for all $x \in (-\infty, -1)$.

For $\lambda = 2$ we can use the same idea. Defining $h\colon \mathbb{R}^+ \to \mathbb{R}$ now by

$$h(x) = 1 + \frac{1}{\psi_2(-1-x)} = \frac{\log((x+1)/x)}{1 - x\log((x+1)/x)},$$

$$h'(x) = \frac{[\log(1+1/x)]^2 - 1/x(1+x)}{[1 - x\log(1+1/x)]^2}.$$

and

$$g(s) := \log(1+s) - \frac{s}{\sqrt{1+s}}, \qquad s \geqslant 0,$$

the sign of $h'(x)$ coincides with the sign of $g(1/x)$. Since

$$g'(s) = \frac{1}{1+s} - \frac{s+2}{2\sqrt{(1+s)^3}} = \frac{-s^2}{2\sqrt{(1+s)^3}\,(2\sqrt{1+s} + s + 2)} < 0$$

and $g(0) = 0$, we obtain $g(s) < 0$ for all $s \in \mathbb{R}_+$ which implies $h'(x) < 0$ for all $x \in \mathbb{R}_+$ and therefore, $\psi'_\lambda(x) < 0$ for all $x \in (-\infty, -1)$. ∎

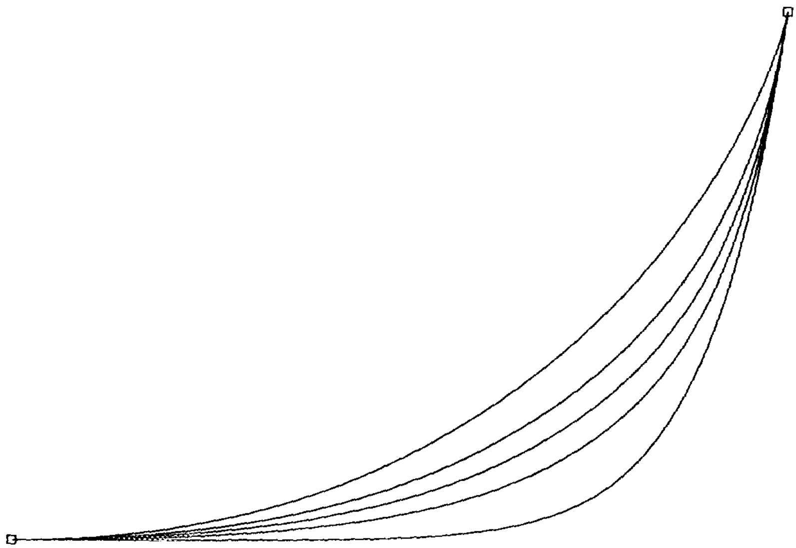

FIGURE 5.1

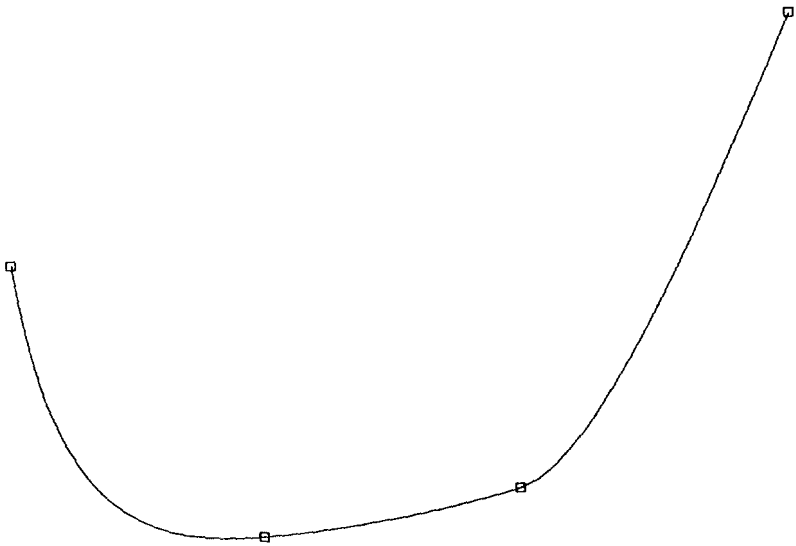

FIGURE 5.2

The previous theorem allows us to define a whole one-parameter family of K^1 methods P_λ, $\lambda \in [1, 2]$ enjoying the properties mentioned in Section 3 by defining $P_\lambda(u \mid x_0, x_1)$ as the solution of the boundary value problem (5.1). Figure 5.1 shows the graphs of $P_\lambda(0, 1, 1, 9)$ for the values of $\lambda = 1, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, 2$. The solution is greater for increasing values of the parameter λ.

In general, low values of λ produce solutions which present rapid variations whereas for high values of λ the solution looks smoother.

As a final example, Fig. 5.2 shows the graph of the solution for the Hermite problem with complete data

x	0	1	2	3
y	10	0	2	20
m	-50	1	3	25

obtained with the local method $P_{5/4}$.

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